7.5a wavelets

Tuesday, February 25, 2020 9:09 AM

Wavelets

Want an orthonormal basis set of the vector space of functions.

Ex. Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i \left(\frac{n}{T}\right) x} \quad \text{for} \quad x \in \left[-\frac{T}{2}, \frac{T}{2}\right]$$

Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi$$

But sines and cosines are distributed in support, so want something with finite support that's also efficiently computable.

Define A **dilation** is a mapping that scales all distances by the same factor.

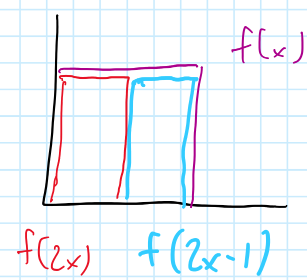
A **dilation equation** is an equation where a function is defined in terms of shifted, scaled versions of itself.

Ex. $f(x) = \sum_{k=0}^{d-1} c_k f(2x-k)$.

Ex. $f(x) = f(2x) + f(2x-1)$

One solution:

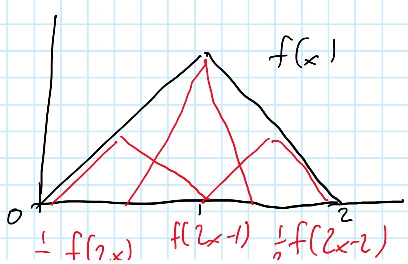
$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$



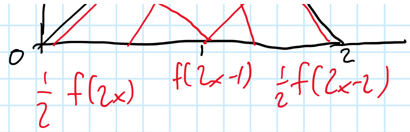
Ex. $f(x) = \frac{1}{2} f(2x) + f(2x-1) + \frac{1}{2} f(2x-2)$

One solution:

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x+1, & 1 \le x < 2 \\ 0, & \text{elsewhere} \end{cases}$$



$\quad 0, \quad$ elsewhere



If a dilation equation is of the form $\sum_{k=1}^{d-1} c_k f(2x-k)$, then we say that all dilations in the equation are factor of 2 reductions.

Lemma 11.1: If a dilation equation in which all dilations are a factor of two reduction, then either the coeffcients on the RHS sum to 2, or the integral $\int_{-\infty}^{\infty} f(x)\,dx = 0$, where $f(x)$ is the solution.

proof.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \sum_{k=0}^{d-1} c_k f(2x-k)\,dx$$

$$= \sum_{k=0}^{d-1} \int_{-\infty}^{\infty} c_k f(2x-k)\,dx \qquad \text{(allowed if 1-norm of function is finite)}$$

$$= \sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(2x)\,dx \qquad \text{(because integrating over entire real line, so shifts don't matter)}$$

$$= \frac{1}{2} \sum_{k=0}^{d-1} c_k \int_{-\infty}^{\infty} f(x)\,dx.$$

$$\Rightarrow \underbrace{\int_{-\infty}^{\infty} f(x)\,dx = 0}_{} \quad \text{or} \quad \underbrace{\sum_{k=0}^{d-1} c_k = 2.}_{}$$

both are allowed and give nonzero soln of fxnes.

Haar wavelet

Let $\underbrace{\phi(x)}_{\text{scale function}}$ be a solution to $f(x) = f(2x) + f(2x-1)$. e.g. $\phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$

Let $\phi_{jk}(x) = 2^{\frac{j}{2}} \phi(2^j x - k)$

scale function

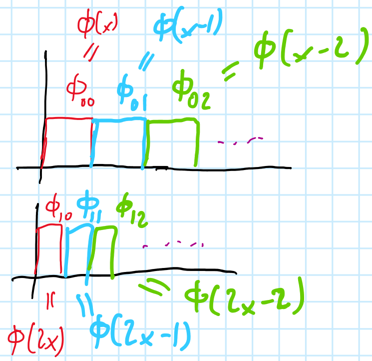
Let $\phi_{jk}(x) = 2^{\frac{j}{2}} \phi(2^j x - k)$

needed for orthonormal basis, but we are going to ignore it for ease of notation.

Let $V_j = \text{span}\{\phi_{jk} \mid k \in \mathbb{N}\}$

$V_0 = \text{span}\{\phi_{00}, \phi_{01}, \phi_{02}, \dots\}$

$V_1 = \text{span}\{\phi_{10}, \phi_{11}, \phi_{12}, \dots\}$



Note $V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_j \subseteq V_{j+1} \subseteq \cdots$ ← this fact is guaranteed because ϕ is the solution to a dilation equation

Recall: We can define orthogonality w.r.t. any inner product.

Let $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dx$.

Then f and g are orthogonal $\iff \langle f, g \rangle = 0$.

orthonormal $\iff \langle f, g \rangle = 0, \ \langle f, f \rangle = 1, \ \langle g, g \rangle = 1$.

Note: $\langle \phi_{jk}, \phi_{j\ell} \rangle = 0 \quad \forall j \neq \ell$ because they have different supports.

So $\{\phi_{j0}, \phi_{j1}, \dots\}$ is a basis of V_j.

$\text{Span}\{\phi_{jk}\}_{j,k}$ = space of functions, but is not a Schauder basis since not linearly ind.

We want to construct a basis out of these functions.

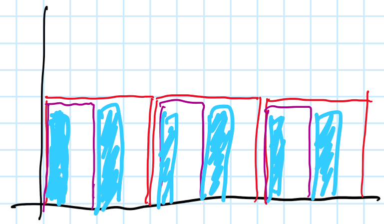
But $\phi_{jk} = \phi_{j+1, 2k} + \phi_{j+1, 2k+1}$

We could in theory delete ϕ_{jk}, but that doesn't work since we're left with only "$\phi_{\infty, k}$", which is not an interesting basis set.

Delete $\phi_{j+1, 2k+1}$ from our set of functions, $\forall k$.

i.e. we are removing all $\phi_{j,k}$, where $j>0$ and k is odd

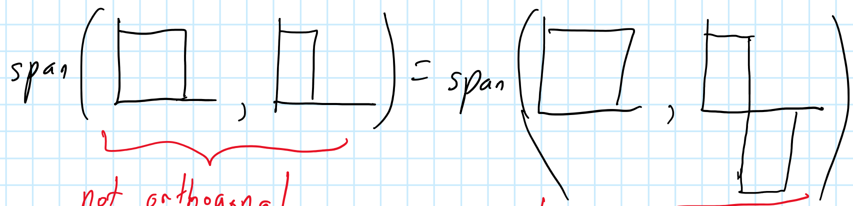
Then we get $\{\phi_{00}, \phi_{01}, \phi_{02}, \ldots$
$\phi_{10}, \phi_{12}, \phi_{14}, \ldots$
$\phi_{20}, \phi_{22}, \phi_{24}, \ldots \}$,



which form a basis for the space of functions.

But $\langle \phi_{jk}, \phi_{j+1,2k} \rangle \neq 0$, so we don't have an orthogonal basis.

Instead, note



span(,) = span(,)

not orthogonal

orthogonal

Thus, let

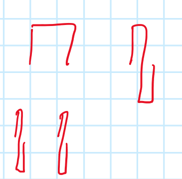
$$\psi_{jk}(x) = \begin{cases} 1, & \frac{2k}{2^j} \le x < \frac{2k+1}{2^j} \\ -1, & \frac{2k+1}{2^j} \le x < \frac{2k+2}{2^j} \\ 0, & \text{otherwise.} \end{cases}$$

And replace $\phi_{j+1,2k}$ with $\psi_{j+1,2k}$, so we

get an orthogonal basis $\{\phi_{00}, \phi_{01}, \phi_{02}, \ldots$
$\psi_{10}, \psi_{12}, \psi_{14}, \ldots$
$\psi_{20}, \psi_{22}, \psi_{24}, \ldots$
$\vdots$ $\}$

Furthermore, we have a basis for all functions supported on the unit interval

$\{\phi_{00}, \psi_{10}$ — support length 1
ψ_{20}, ψ_{22} — support length $\frac{1}{2}$
$\psi_{30}, \psi_{32}, \psi_{34}, \psi_{36}$ — support length $\frac{1}{4}$
$\psi_{40}, \psi_{42}, \ldots, \psi_{4,14}$ — support length $\frac{1}{8}$
$\vdots$



$\psi_{40}, \psi_{42}, \ldots, \psi_{4,14}$ support length $\frac{1}{8}$

$\vdots$ }

For any finite support function, we can approximate it by choosing a scale vector $\phi(x)$ whose scale is that of the support of the function.

It is straight-forward to approximate it with $\phi_{jk}(x)$ for fixed j

(getting a 2^j-point sample)

$f(x) \approx \sum_{k=0}^{2^j-1} s_k \phi(2^j x - k)$, which we can write as $(s_0, s_1, \ldots, s_{2^j-1})$.

To rewrite in the Haar basis we defined over the unit interval, need to find c_i's

$$\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{pmatrix} = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \\ \begin{array}{cccccccc} \phi_{00} & \psi_{10} & \psi_{20} & \psi_{22} & \psi_{30} & \psi_{32} & \psi_{34} & \psi_{36} \end{array} \end{array} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_7 \end{pmatrix}$$

Transform the basis into the Haar basis from the evenly spaced basis.

Matrix inverses are slow, but here we can do better.

$$s_0 - s_1 / 2 = c_4 \qquad s_2 - s_3 / 2 = c_5 \qquad s_4 - s_5 / 2 = c_6 \qquad s_6 - s_7 / 2 = c_7$$

$$\frac{s_0+s_1}{2} - \frac{s_2+s_3}{2} / 2 = c_2 \qquad \frac{s_4+s_5}{2} - \frac{s_6+s_7}{2} / 2 = c_3$$

$$\frac{s_0+s_1+s_2+s_3}{4} - \frac{s_4+s_5+s_6+s_7}{4} / 2 = c_1$$

$$\frac{s_0+\cdots+s_7}{8} = c_0$$